ADMISSIBILITY OF THREE SATTERTHWAITE APPROXIMATE F-STATISTICS IN ONE PRELIMINARY TEST PROCEDURES IN A MIXED MODEL

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SUMMARY

Besides size and power, admissibility of a test procedure is a much desired property. It has been proved that Satterthwaite approximate F-statistics in one preliminary test procedures in a mixed model are admissible. Necesssary and sufficient conditions for admissibility have been derived.

INTRODUCTION

The expected mean squares, in an ANOVA table for different factors of a factorial experiment, often reveal that for testing hypothesis about certain main effect(s) no mean square is adequate to be used as error mean square unless one or more interactions are zero. Therefore, it becomes logical to test the presence of the doubtful interaction(s) prior to the testing of main hypothesis. Such tests in literature are termed as tests based on conditional specification. For a detailed bibliography on inferences based on conditional specification *vis-a-vis* on preliminary test of significance (PTS), see Bancroft and Han [1]. In cases, where these interactions do not turn out to be non-significant, the use of Satterthwaite approximate F-statistics [9] can be made to test the main hypothesis.

So far, the size and power of the test procedures have been the sole criterion for selection of the tests. Besides these two, it is also equally importants to find out whether a test procedure is admissible or not. The reason for this is that inadmissibility of a test procedure is often a compelling reason for rejecting it. Cohen [3] derived the condition for inadmissibility of a test procedure involving one PTS in in a random effect model. In this paper, the authors have shown that under certain restrictions, test procedures using Satterthwaite approximate F-statistics are also admissible.

Enunciation of Problem

Consider a three factor factorial experiment with factors A, B and C at levels a, b and c respectively arranged in a randomized block design with r blocks. A is taken as fixed effect while factors B and C are taken as random. For this the appropriate statistical model is,

$$X_{ijkl}=\mu+\alpha_i+\beta_j+\gamma_l+\rho_K+(\alpha\beta)_{ij}+(\alpha\gamma)_{il}+(\beta\gamma)_{jl}+(\alpha\beta\gamma)_{ijl}+\varepsilon_{ijlk} \quad \ldots(2.1)$$

where $i=1, 2,\ldots, a,$

$j=1, 2,\ldots, b;\ l=1, 2, \ldots c$

and $k=1, 2,\ldots, r.$

Also $\sum_i \alpha_i=\sum_i (\alpha\beta)_{ij}=\sum_i (\alpha\gamma)_{il}=0,$

ε_{ijlk} are IID $N(0,\ \sigma_e^2)$

whereas $\sum_i (\alpha\beta)_{ij}$

and $\sum_i (\alpha\gamma)_{il}$ are not assumed zero.

Abridged ANOVA table for 3-factor factorial experiment for mixed model

Source	*d.f.*	*M.S.*	*E.M.S.*
Effect A	$(a-1)=n_4$	V_4	$rbc \sum_i \alpha_i^2/(a-1)+rc\,\sigma_{AB}^2+rb\,\sigma_{AC}^2+r\,\sigma^2_{ABC}+\sigma_e^2=\sigma_4^2$
Doubtful error AB	$(a-1)(b-1)=n_3$	V_3	$rc\,\sigma_{AB}^2+r\,\sigma_{ABC}^2+\sigma_e^2=\sigma_3^2$
AC	$(a-1)(c-1)=n_2$	V_2	$rb\,\sigma_{AC}^2+r\,\sigma_{ABC}^2+\sigma_e^2=\sigma_2^2$
Error ABC	$(a-1)(b-1)(c-1)=n_1$	V_1	$r\,\sigma_{ABC}^2+\sigma_e^2=\sigma_1^2$.

The hypothesis under the test is $H_o : \alpha_i = 0$ *vs* $H'_o : \alpha_i > 0$. It is obvious from the ANOVA table that unless the interaction $\sigma^2_{AB} = 0$, no appropriate expected mean square for testing H_o is available which can be used as error. Therefore, the first test $H_{o1} : \sigma_3^2 = \sigma_1^2$ *vs* $H'_{o1} : \sigma_3^2 > \sigma_1^2$. In case H_{o1} is rejected, three test procedures using Satterthwaite approximate F-Statistics have been developed and their admissibility have been worked out. Each test procedures, consists of two steps.

Procedures—1 :

Step1 : $\dfrac{V_3}{V_1} > \beta_1 ; \dfrac{V_4 + V_1}{V_3 + V_2} > \beta_3$ ...(2.2)

Step 2 : $\dfrac{V_3}{V_1} \leqslant \beta_1 ; \dfrac{V_4}{V_2} > \beta_2$

Procedure II :

Step 1 : $\dfrac{V_3}{V_1} > \beta_1 ;\ V_4/(V_3 + V_2 - V_1) > \beta_{32}$

Step 2 : $\dfrac{V_3}{V_1} \leqslant \beta_1 ; \dfrac{V_4}{V_2} > \beta_2$ ...(2.3)

Procedure III :

Step 1 : $\dfrac{V_3}{V_1} > \beta_1 ; \dfrac{V_4 - V_3}{V_2 - V_1} > \beta_{33}$

Step 2 : $\dfrac{V_3}{V_1} \leqslant \beta_1 ; \dfrac{V_4}{V_2} > \beta_2$ ...(2.4)

where
$$\beta_1 = F(n_3, n_1, \alpha_1);$$
$$\beta_3 = F(\nu_1, \nu_2, \alpha_3);$$
$$\beta_2 = F(\nu, n_2, \alpha_2)$$
$$\beta_{32} = F(\nu, \nu_3, \alpha_3);$$
$$\beta_{33} = F(\nu_4, \nu_5, \alpha_3).$$

We know that $n_i V_i / \sigma_i^2$ is distributed as central chi-square with n_i *d.f.* ($i = 1, 2, 3$) and using Patnaik's approximation [7] $(n_4 v_4)/(\sigma_1^2)$ is distributed as central chi-square with ν *d.f.* where,

$$\nu = n_4 + 4\ \lambda^2/(n_4 + 4\lambda);$$
$$\lambda = n_4\,(\theta_{41} - 1)/2;\ c = 2 - \theta_{14}.$$

Degrees of freedom ν_t $(t=1, 2, 3, 4, 5)$ are formulated by Satterthwaites' approach [8] and we obtain,

$$\nu_1 = (\nu c n_4^{-1}+1)^2/(\nu c^2 n_4^{-2}+n^{-1});$$

$$\nu_2 = (\theta_{12}+\theta_{13})^2/(\theta^2_{12} n_3^{-1}+\theta^2_{13} n_2^{-1}),$$

$$\nu_3 = (\theta_{12}^{-1}+\theta_{13}^{-1}-1)^2 / (\theta_{12}^{-2}\, n_2^{-1}\, \theta_{13}^{-2}\, n_3^{-1} + n_1^{-1}),$$

$$\nu_4 = (\nu c\, \theta_{13}\, n_4^{-1}-1)^2 / (c^2 \nu\, \theta_{13}^2\, n_4^{-2}+n_1^{-1}), \qquad \ldots(2.5)$$

and $$\nu_5 = (\theta_{12}^{-1}-1)^2 / (\theta_{12}^{-2}\, n_2^{-1} + n_1^{-1})$$

and $$\theta_{ij} = \sigma_i^2 / \sigma_j^2 \text{ for } i \neq j.$$

In the above, α_1 is called the preliminary level of significance. ν is estimated for a valve of λ depending on θ_{14} as n_4 is fixed in an experiment. ν_t may be in fractions. Values of F for fractional *d.f.* can be interpolated by formulae given by Laubscher [5] or may be read from Mardia and Zenroch [6].

Conditions of Admissibility

A test procedure δ for testing a hypothesis about the parameter θ is generally rejected if there exists a procedure δ' such that the risk function,

$$R(\theta, \delta') \leqslant R(\theta, \delta) \text{ for all } \theta$$

and $$R(\theta, \delta') < R(\theta, \delta) \text{ for some } \theta \qquad \ldots(3.1)$$

In such a situation δ is inadmissible and conversely, δ is admissible. Denoting the test procedure by $\psi(V)$, its value is unity under each step of the test procedure (2.2), (2.3) or (2.4) if it stands true otherwise zero. Davenport and Webster [4] considered only the size and power of the three approximate F-tests. We derive below necessary and sufficient conditions for the three test procedures (described earlier) to be admissible. A test procedure $\psi(V)$ is admissible if and only if the acceptance region of $\psi(V)$ have convex section in V_4 for fixed (V_1, V_2, V_3) or otherwise we prove that the sections of the critical region in V_4 for given (V_1, V_2, V_3) are half lines.

Theorem 1. Necessary and sufficient condition for the test procedure-1 to be admissible is,

$$=\frac{(\beta_1\beta_3+\beta_1+\beta_3-3)}{\{3\beta_1\beta_3-(\beta_1\beta_3+\beta_1-1)\beta_2-2\}}$$

$$>\frac{\{\beta_3(4\beta_2+\beta_1-2)+(\beta_1-2)-\beta_2^2\,(\beta_1\beta_3+\beta_1+2\beta_3-2)\}}{(1-\beta_2)\{\beta_3(2\beta_1-\beta_2+2)-\beta_1(\beta_3\beta_2+\beta_2+1)\}} \quad \ldots(3.2)$$

The joint p.d.f. of V_1, V_2, V_3 and V_4 after making an orthogonal transformation,

$$W=T\,V$$

where $W'=(W_4, W_2, W_3, W_1)$

and $V'=(V_1, V_2, V_3, V_4)$

and

$$T=\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

can be expressed as,

$$dP(W,\theta)=C(\theta)\ \exp.\ (W'\theta)\ d\lambda(W)$$

where $C(\theta)=\mathrm{K}'$,

$$\theta=T(n_1/\sigma_1^2,\ n_2/\sigma_2^2,\ n_3/\sigma_3^2,\ n_4C^{-1}/\sigma_1^2)$$

and $d\lambda(W)$ is the function of W's and differential terms. It is trivial to show that the conditional distribution of W_4 given (W_1, W_2, W_3) belongs to the family of exponential distribution with parameter θ_4 (say).

The admissibility of the test procedure $\phi(W)$ can be proved with the help of basic principle {see Cohen [3]}. To prove $\phi(W)$ to be admissible, it is shown that the acceptance region in W_4 for fixed W_1, W_2 and W_3 have convex section.

Substituting for V_i's ($i=1, 2, 3, 4$) in terms of W_i's, solitany test in Procedure-I leads to the following inequalities :

$$\frac{V_3}{V_1}>\beta_1 \Rightarrow W_4>2\{-(1+\beta_1)W_2/2\sqrt{3}-W_3/\sqrt{6}+W_1/\sqrt{2}\}/(1-\beta_1) \quad \ldots(3.3)$$

$$\frac{V_4}{V_2} > \beta_2 \Rightarrow W_4 > 2\{(\beta_2-1)W_2/2\sqrt{3}-(1+2\beta_2)W_3/\sqrt{6}$$
$$-W_1/\sqrt{2}\}/(1-\beta_2) \quad \ldots(3.4)$$

$$(V_4+V_1)/(V_3+V_2) > \beta_3 \Rightarrow W_4 > (W_2/\sqrt{3}-W_3/\sqrt{6}$$
$$-W_1/\sqrt{2})(1+\beta_3)/(1-\beta_3) \quad \ldots(3.5)$$

Let the right hand side of the inequalities (3.3) to (3.5) be denoted by E_1, E_2 and E_3 respectively. Cohen [2] showed that E_i's ($i=1, 2, 3$) are spheres with centres at the orign.

Acceptance region under procedure-I will be given by the union of sets,

$$W_4 : W_4 \leqslant E_3 \cap W_4 > E_1$$
$$W_4 : W_4 \leqslant \min (E_1, E_2) \quad \ldots(3.6)$$

If $E_1 < E_3 < E_2$ hold, the accentance region given by the union of the sets (3.6) is a convex set.

Eliminating W_1, W_2 and W_3 from the relations $E_1 < E_3$, $E_1 < E$ and $E_3 < E_2$, the condition (3.2) is readily obtained.

Theorem 2. The test procedure $\psi(V)$ given by (2.3) is admissible if and only if,

$$\frac{\{2\beta_{32}(2-\beta_2)+2\beta_1(\beta_2\beta_{32}-\beta_{32}+2\beta_2)-\beta_1^2\,\beta_2(\beta_{32}+1)\}}{\{2\beta_{32}(2-\beta_2)+\beta_1(2\beta_2\beta_{32}-8\beta_{32}+\beta_2)+\beta_1^2\,(4\beta_{32}-\beta_2\beta_{32}-\beta_2\}}$$
$$< \frac{(\beta_2+1)\{\beta_1(1+\beta_{32})-2\beta_{32}\}}{\{\beta_1(\beta_2\beta_{32}+\beta_2-2)-2\beta_{32}(2-\beta_1-\beta_2)\}} \quad \ldots(3.7)$$

The proof is similar to that of the theorem 1, whereas E_3 will be

$$E_3 = 2\{\frac{W_2}{2\sqrt{3}}(5\beta_{32}-1)-\frac{W_3}{\sqrt{6}}(1+\beta_{32})-\frac{W_1}{\sqrt{2}}(1+\beta_{32})\}/(1-\beta_{32})$$

Remarks. (*i*) There can be in all six inequality relations among E_1, E_2 and E_3.

(*ii*) When relations $E_1 < E_2 < E_3$, $E_3 < E_1 < E_2$ and $E_1 < E_3 < E_2$ hold, procedures I and II are admissible where as for relations $E_2 < E_1 < E_3$, $E_2 < E_3 < E_1$ and $E_3 < E_2 < E_1$, the acceptance region is not a convex set and hence the said procedures are inadmissible.

(*iii*) Necessary and sufficient conditions for the relations $E_3 < E_1 < E_2$ and E_1, $E_2 < E_3$ can be derived in the same manner as for (3.2).

(*iv*) α_1 is the preliminary level of significance and can arbitrarily be chosen in such a way that condition (3.2) is satisfied for procedure I and (3.7) for procedure II.

(*v*) Among the class of admissible test procedures, one must be chosen having the maximum power of the test.

Procedure III. Taking into consideration, the first step of this procedure, $\frac{V_4-V_3}{V_2-V_1} \geqslant \beta_{33}$ leads to the relation,

$$0 > \left(\frac{W_2}{\sqrt{3}} - \frac{W_3}{\sqrt{6}}\right)\beta_{33} - \frac{W_1}{\sqrt{2}} \qquad \ldots(3.8)$$

In such a situation E_3 is taken to be zero.

Acceptance region will be given by the union of the sets,

$$W_4 : W_4 > E_1$$
$$W_4 : W_1 \leqslant \min(E_1, E_2) \qquad \ldots(3.9)$$

when $E_1 < E_2$, procedure-III is admissible and otherwise inadmissible. But this is one of the peculiar situations arising out in developing the condition for a test procedure to be admissible. We have only one inequality $E_1 < E_2$ involving W_1, W_2 and W_3 which need to be eliminated to get a condition purely in betas which does not appear to be possible. The only feasibility for upholding the inequality $E_1 < E_2$ is to consider the corresponding coefficient of W's and to choose α_1 suitably.

It may be further remarked that the test statistics as the ratio of linear combination of variances having negative coefficient are not recommended from the point of view of power of test as stated by Davenport and Webster [4]. Since such a procedure in one preliminary test does not lead to a clear condition for admissibility, its use should be avoided as far as possible from admissibility point of view also.

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